Theory of Equations

1. Let $f(x) = x^4 - 5x^3 + 4x^2 + 8x - 8$, $1+\sqrt{5}$ is a root of $f(x) = 0$ $\Rightarrow$ $1-\sqrt{5}$ is a root of $f(x) = 0$.

$\therefore$ $\left[x-\left(1+\sqrt{5}\right)\right]\left[x-\left(1+\sqrt{5}\right)\right] = (x-1)^2 - 5 = x^2 - 2x - 4$ is a factor of $f(x)$.

By division, $f(x) = (x^2 - 2x - 4)(x^2 - 3x + 2) = \left[x-\left(1+\sqrt{5}\right)\right]\left[x-\left(1+\sqrt{5}\right)\right][x-2][x-1]$.

The roots of $f(x) = 0$ are $1, 2,$ $1\pm\sqrt{5}$.

2. Let $f(z) = z^4 - 4z^2 + 8z - 4$, $1 + i$ is a root of $f(z) = 0 \Rightarrow 1 - i$ is also is a root of $f(z) = 0$.

$\therefore$ $[z - (1 + i)]\,[z - (1 - i)] = (z - 1)^2 - i^2 = z^2 - 2z + 2$ is also is a factor of $f(z)$.

By division, $f(z) = (z^2 - 2z + 2)(z^2 + 2z - 2) = [z - (1 + i)]\,[z - (1 - i)]\left[z-\left(-1+\sqrt{3}\right)\right]\left[z-\left(-1-\sqrt{3}\right)\right]$

The roots of $f(z) = 0$ are $1\pm i$, $-1\pm\sqrt{3}$.

3. Let $f(x) \equiv A\,x\,(x - 1)(x + 2) + Bx(x - 1) + Cx + D$

$f(0) = -2$ $\Rightarrow$ $D = -2$ …. (1)

$f(1) = -6$ $\Rightarrow$ $C + D = -6$ …. (2) ; (1) $\downarrow$(2), $C = -4$ …. (3)

$f(-2) = -12$ $\Rightarrow$ $6B - 2C + D = -12$ …. (4) ; (1),(2) $\downarrow$(3), $B = -3$ …. (5)

$\therefore$ $f(x) \equiv A\,x\,(x - 1)(x + 2) - 3x(x - 1) - 4x - 2 \equiv A\,x^3 + (A - 3)x^2 - (2A + 1)x - 2$ …. (6)

Product of roots $= -(-2)/A = 2/A$, Sum of roots $= -(A - 3)/A$

Product of its roots is twice their sum $\Rightarrow$ $2/A = 2\,[-(A - 3)/A]$ $\Rightarrow$ $A = 2$

From (6), $f(x) = 2x^3 - x^2 - 5x - 2$.

$f(-1) = 0 \Rightarrow$ $(x + 1)$ is a factor of $f(x)$ by Factor Theorem

$f(x) = (x + 1)(2x^2 - 3x - 2) = (x + 1)(x - 2)(2x + 1)$

$\therefore$ The roots of $f(x) = 0$ are $x = -1$ or 2 or $-1/2$.

4. Let α, β, γ be the roots of $x^3 - 5x^2 - 16x + 80 = 0$

$\therefore$ $\alpha + \beta + \gamma = 5$ …. (1) But $\alpha + \beta = 0$ …. (2)

(2) $\downarrow$ (1), $\gamma = 5$ …. (3) Also $\alpha\beta\gamma = -80$ …. (4)

(3) $\downarrow$ (4), $\alpha\beta = -16$ …. (5) By (2) & (5), $\alpha = 4$, $\beta = -4$

$\therefore$ The roots are $4, -4, 5$.

5. Let $f(x) = x^4 - 12x^2 + 12x - 3$

$f(-3) = -66 < 0$, $f(-4) = 13 > 0$ There is a sign change in $f(x)$ as x change from -4 to -3, therefore $f(x) = 0$ has a root between -3 and -4.

$f(2) = -11 < 0$, $f(3) = 6 > 0$. There is a sign change in $f(x)$ as x change from 2 to 3, therefore $f(x) = 0$ has a root between 2 and 3.

6. a , b are the real roots of the equation $x^2 - mx + n = 0$,

$\therefore$ $a + b = m$ …. (1) ; $ab = n$ …. (2)

The equation $2x^2 - qx + r = 0$ has roots $a + 1$, $b + 2$,

$\therefore$ $(a + 1)(b + 2) = q/2 \quad \Rightarrow \quad a + b = q/2 - 3$ …. (3)

$(a + 1)(b + 2) = r/2 \quad \Rightarrow \quad ab + 2a + b = r/2 - 2$ …. (4)

(1) $\downarrow$ (2), $\therefore$ $m = q/2 - 3 \quad \Rightarrow \quad q = 2m + 6$ …. (5)

$x^2 - mx + n = 0 \quad \Rightarrow \quad x = \dfrac{m \pm \sqrt{m^2 - 4n}}{2} \quad \Rightarrow \quad a = \dfrac{m + \sqrt{m^2 - 4n}}{2}$ …. (6)

(1), (2) $\downarrow$ (4), $n + m + a = r/2 - 2 \quad \Rightarrow \quad m + n + \dfrac{m + \sqrt{m^2 - 4n}}{2} = \dfrac{r}{2} - 2$ …. (7)

$\therefore$ $r = 3m + 2n + 4 + \sqrt{m^2 - 4n}$ …. (8)

If $a = b$, (1) and (2) becomes $m = 2a$ …. (9) and $n = a^2$ …. (10)

(9), (10) $\downarrow$ (5), (8) $q = 4a + 6$ and $r = 6a + 2a^2 + 4$

Hence, $q^2 = (4a + 6)^2 = 4(2a + 3)^2 = 4(4a^2 + 12a + 9) = 4[2(2a^2 + 6a + 4) + 1] = 4(2r + 1)$

7. By putting $y = \dfrac{x^2}{x+1}$ …. (1), $\dfrac{x^2}{x+1} + \dfrac{2x+2}{x^2} - 3 = 0$ becomes $y + \dfrac{2}{y} - 3 = 0$

or $y^2 - 3y + 2 = 0 \quad \Rightarrow \quad (y - 1)(y - 2) = 0 \quad \Rightarrow \quad y = 1$ or 2.

By (1), $\dfrac{x^2}{x+1} = 1$ or $\dfrac{x^2}{x+1} = 2$

$x^2 - x - 1 = 0$ or $x^2 - 2x - 2 = 0$

$x = \dfrac{1 \pm \sqrt{5}}{2}$ or $1 \pm \sqrt{3}$.

8. $ax^2 + by^2 = 1$ …. (1) $Lx + My = 1$ …. (2)

From (2), $y = \dfrac{1 - Lx}{M}$ …. (3) (3) $\downarrow$ (1), $ax^2 + b\left(\dfrac{1 - Lx}{M}\right)^2 = 1$

Simplify, we get $x^2(aM^2 + bL^2) - 2Lbx + b - M^2 = 0$ …. (4)

Since there is only one distinct root in (4), $\therefore$ $\Delta = 0$.

$\therefore$ $(-2Lb)^2 - 4(aM^2 + bL^2)(b^2 - M^2) = 0 \quad \Rightarrow \quad abM^2 - aM^4 - M^2 b^2 L^2 = 0$

Since $M \neq 0$, we get $ab - aM^2 - bL^2 = 0$ $\quad \therefore \quad \dfrac{L^2}{a} + \dfrac{M^2}{b} = 1$.

From (4), since $\Delta = 0$,

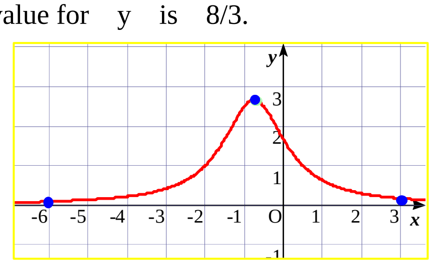
$$x = \frac{2Lb \pm \sqrt{0}}{2(aM^2 + bL^2)} = \frac{bL}{aM^2 + bL^2} \quad ; \quad y = \frac{1 - Lx}{M} = \frac{1 - L\left(\dfrac{bL}{aM^2 + bL^2}\right)}{M} = \frac{aM}{aM^2 + bL^2}.$$

9. Let $y = \dfrac{5}{2x^2 + 3x + 3}$, $(2x^2 + 3x + 3)\, y = 5$, $(2y)x^2 + (3y)x + (3y - 5) = 0$

Since x is real, $\therefore \Delta \geq 0 \quad \Rightarrow \quad (3y)^2 - 4(2y)(3y - 5) \geq 0 \quad \Rightarrow \quad y(3y - 8) \leq 0 \quad \Rightarrow \quad 0 \leq y \leq 8/3$.

$\therefore$ y is positive for all real values of x and the greatest value for y is $8/3$.

The sketch is shown on the right.



Max. point = $\left(-\frac{3}{4}, \frac{8}{3}\right)$

Two end points = $\left(-6, \frac{5}{57}\right)$, $\left(3, \frac{1}{6}\right)$

10. If α, β are the roots of $x^2 + 7x - 3 = 0$, then (1) $\alpha^2 + 7\alpha - 3 = 0$, (2) $\beta^2 + 7\beta - 3 = 0$.

Since $\alpha, \beta \neq 0$, (1) $\times\alpha$ + (2) $\times \beta$, $\alpha^3 + \beta^3 + 7(\alpha^2 + \beta^2) - 3(\alpha + \beta) = 0$.

11. Let the roots of $ax^2 + bx + c = 0$ be $p\alpha$ and $q\alpha$.

$\therefore$ $p\alpha + q\alpha = -b/a$ …. (1) $(p\alpha)(q\alpha) = c/a$ …. (2)

From (1), $a^2(p + q)^2 \alpha^2 = b^2$ …. (3) From (2), $a(pq)\alpha^2 = c^2$ …. (4)

(3)/(4), $\frac{(p+q)^2}{pq} = \frac{b^2}{ac}$ $\Rightarrow$ $ac(p + q)^2 = b^2 pq$.

12. $y = x - x^{-1}$ …. (1) $3(x - x^{-1}) = x^2 + x^{-2}$ …. (2)

(1) $\downarrow$ (2), $3y = y^2 - 2$ $\Rightarrow$ $y^2 - 3y + 2 = 0$ $\Rightarrow$ $(y - 1)(y - 2) = 0$ $\Rightarrow$ $y = 1$ or 2

From (1), $x - x^{-1} = 1$ or $x - x^{-1} = 2$

$x^2 - x - 1 = 0$ or $x^2 - 2x - 1 = 0$

$x = \frac{1 \pm \sqrt{5}}{2}$ or $x = 1 \pm \sqrt{2}$

13. If the roots of $ax^2 + bx + c = 0$ are α, β, then $\alpha + \beta = -b/a$, $\alpha\beta = c/a$ …. (1)

$$\alpha - \beta = \pm\sqrt{(\alpha-\beta)^2} = \pm\sqrt{(\alpha+\beta)^2 - 4\alpha\beta} = \pm\sqrt{\left(-\frac{b}{a}\right)^2 - 4\left(\frac{c}{a}\right)} = \frac{\pm\sqrt{b^2 - 4ac}}{a} \quad …. \ (2)$$

$$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha+\beta}{\alpha\beta} = \frac{-b/a}{c/a} = -\frac{b}{c} \quad …. \ (3)$$

$$\alpha - \beta = \frac{1}{2}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) \Rightarrow \frac{\pm\sqrt{b^2 - 4ac}}{a} = \frac{1}{2}\left(-\frac{b}{c}\right) \Rightarrow \frac{b^2 - 4ac}{a^2} = \frac{b^2}{4c^2} \quad \Rightarrow \quad b^2(4c^2 - a^2) = 16\,ac^3 .$$

14. The roots of $px^2 + x + q = 0$ are α, β . $\alpha + \beta = -1/p$, $\alpha\beta = q/p$ …. (1)

$(p\alpha^3 + q\alpha) + (p\beta^3 + q\beta) = \alpha(p\alpha^2 + \alpha + q) + \beta(p\beta^2 + \beta + q) + 2\alpha\beta - (\alpha + \beta)^2$

$= \alpha(0) + \beta(0) + 2(q/p) - (-1/p)^2 = \frac{2q}{p} - \frac{1}{p^2}$

15. **(Method 1)**

$x^2 + y^2 = 1$ …. (1) , $Lx + My = 1$ …. (2)

From (2), $y = (1 - Lx)/M$ …. (3)

(3)$\downarrow$(2), $x^2 + \left(\frac{1 - Lx}{M}\right)^2 = 1$ $\Rightarrow$ $(M^2 + L^2)x^2 - 2Lx + (1 - M^2) = 0$

For real value of x, $\Delta \geq 0$ $\Rightarrow$ $(2L)^2 - 4(M^2 + L^2)(1 - M^2) \geq 0$

$\Rightarrow$ $4M^2(M^2 + L^2 - 1) \geq 0$

Since $M^2 \geq 0$ $\therefore$ $M^2 + L^2 - 1 \geq 0$ or $L^2 + M^2 \geq 1$.

(Method 2)

By Cauchy-Buniakowski-Schwartz (CBS) inequality,

$(x^2 + y^2)(L^2 + M^2) \ge Lx + My = 1$

As all the roots are real and $x^2 + y^2 = 1$, $\therefore (L^2 + M^2) \ge 1$.

16. **(a)** $f(x) - k = \dfrac{x^2 + 2ax + b}{x^2 + 1} - k = \dfrac{x^2 + 2ax + b - kx^2 - k}{x^2 + 1} = \dfrac{(1-k)x^2 + 2ax + (b-k)}{x^2+1} = \dfrac{g(x)}{x^2+1}$ …. (1)

$g(x)$ is a perfect square $\Rightarrow \Delta = 0$ for the equation $g(x) = 0$

$\therefore \quad (2a)^2 - 4(1-k)(b-k) = 0 \Rightarrow k^2 - (1+b)k - (a^2 - b) = 0$ …. (2)

$\therefore \quad k_1 = \dfrac{(1+b) - \sqrt{b^2 + 6b + 1 - 4a^2}}{2}, \quad k_2 = \dfrac{(1+b) + \sqrt{b^2 + 6b + 1 - 4a^2}}{2}$ …. (3)

If $\Delta = 0$, the roots (double roots) for $g(x) = 0$ is $x = \dfrac{2a}{2(1-k_r)} = \dfrac{a}{1-k_r}$, $r = 1, 2$.

$\therefore [(1 - k_r)x + a]^2$ is a factor of $g(x)$.

By comparison of coefficient of x^2 –term, $g(x) = \dfrac{[(1-k_r)x + a]^2}{1-k_r}$.

$\therefore \quad f(x) - k_r = \dfrac{g(x)}{x^2+1} = \dfrac{[(1-k_r)x + a]^2}{(1-k_r)(x^2+1)}$, where $r = 1, 2$. …. (4)

(b) From (2), $k_1 + k_2 = 1 + b$, $k_1k_2 = -(a^2 - b)$ …. (5)

$\therefore \quad (1-k_1)(1-k_2) = 1 - (k_1 + k_2) + k_1k_2 = 1 - (1+b) - (a^2 - b) = -a^2$ …. (6)

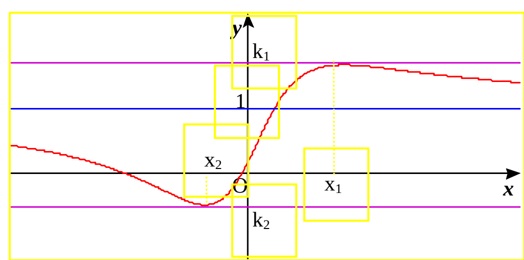
From (6), $(1-k_1)(1-k_2) \le 0$.

Since $k_1 < k_2$, $1 - k_1 \ge 0$ and $1 - k_2 \le 0$ …. (7)

From (4) and (7), $f(x) - k_1 \ge 0$ and $f(x) - k_2 \le 0$ $\Rightarrow$ $k_1 \le f(x) \le k_2$. …. (8)

(c) From (7), $k_1 \le 1 \le k_2$.

When $y = 1$. $1 = \dfrac{x^2 + 2ax + b}{x^2 + 1} \Rightarrow x = \dfrac{1-b}{2a}$



When $y = k_1$, $f(x) - k_1 = 0$, from (4), $x_1 = -\dfrac{a}{1-k_1}$

When $y = k_2$, $f(x) - k_2 = 0$, from (4), $x_2 = \dfrac{a}{k_2 - 1}$.

17. $(a-b)x^2 - 2(a^2 + b^2)x + (a^3 - b^3) = 0$ has real root(s) $\Leftrightarrow$ $\Delta \ge 0$ $\Leftrightarrow$ $4[a^2 + b^2]^2 - 4(a-b)(a^3 - b^3) \ge 0 \Leftrightarrow ab(a+b)^2 \ge 0$ $\Leftrightarrow$ $ab \ge 0$ $\Leftrightarrow$ a and b have the same sign.

Similarly, the equation has complex roots iff a and b have the same sign.

Let α, β be the roots of the given equation. By Vieta's theorem,

$\alpha + \beta = \dfrac{2(a^2 + b^2)}{a - b}, \qquad \alpha\beta = \dfrac{a^3 - b^3}{a - b}$

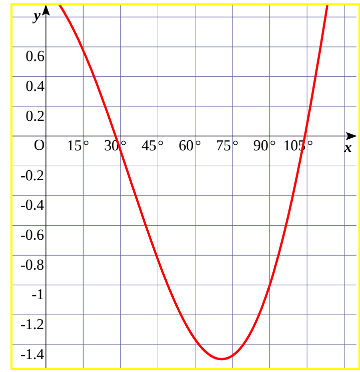
$\alpha - \beta = \sqrt{(\alpha+\beta)^2 - 4\alpha\beta} = \sqrt{\left[\dfrac{2(a^2+b^2)}{a-b}\right]^2 - 4\left(\dfrac{a^3 - b^3}{a-b}\right)} = \dfrac{2(a+b)\sqrt{ab}}{a-b}$

18. If the roots of $ax^2 + bx + c = 0$ are α, β, then $\alpha + \beta = - b/a$, $\alpha\beta = c/a$ .... (1)

Let α', β' be the roots of the equation $x + 2 + \frac{1}{x} = \frac{b^2}{ac} \Leftrightarrow x^2 + \frac{2ac - b^2}{ac}x + 1 = 0$ .... (2)

$$\alpha' + \beta' = -\frac{2ac - b^2}{ac} = -\frac{2(c/a) - (b/a)^2}{a/c} = -\frac{2\alpha\beta - (\alpha+\beta)^2}{\alpha\beta} = \frac{\alpha^2 + \beta^2}{\alpha\beta} = \frac{\alpha}{\beta} + \frac{\beta}{\alpha}$$

$\alpha'\beta' = 1 = \frac{\alpha}{\beta} \times \frac{\beta}{\alpha}$ $\therefore$ The roots of (2) are $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$.



19. $x^2 \cos^2\theta + ax\left(\sqrt{3}\cos\theta + \sin\theta\right) + a^2 = 0$ .... (1)

Since the roots of (1) in x are real, $\Delta \ge 0$,

$\left(a\sqrt{3}\cos\theta + \sin\theta\right)^2 - 4a^2\cos^2\theta \ge 0 \Leftrightarrow$ $4\cos^2\theta - 3\cos\theta - \sin\theta \le 0$

Using graphical method, $28.0^\circ \le \theta \le 104.1^\circ$ approximately.

20. $\alpha : \beta = \lambda : \mu$ $\Rightarrow$ $\alpha = k\lambda$, $\beta = k\mu$

$\alpha + \beta = k\lambda + k\mu = - b/a$ $\Rightarrow$ $k(\lambda + \mu) = - b/a$ $\Rightarrow$ $(\lambda + \mu)^2 ca = b^2c / k^2a$

$\alpha\beta = (k\lambda)(k\mu) = c/a$ $\Rightarrow$ $k^2\lambda\mu = c/a$ $\Rightarrow$ $\lambda\mu b^2 = b^2c / k^2a$

$\therefore$ $\lambda\mu b^2 = (\lambda + \mu)^2 ca$ .... (1)

For $a'x^2 + b'x + c' = 0$, since the roots have the same ratio, $\lambda\mu b'^2 = (\lambda + \mu)^2 c'a'$ .... (2)

(1)/(2) and cross multiply, $a'c'b^2 = acb'^2$.

21. $(1 - a)x^2 + x + a = 0$ .... (1) $0 < a < 1$ .... (2)

Δ of (1) $= 1^2 - 4(1 - a)a = 1 - 4a + 4a^2 = (1 - 2a)^2 \ge 0$ $\Rightarrow$ The roots of (1) are real.

The roots of (1) are $x = \frac{-1 \pm \sqrt{\Delta}}{2(1-a)} = -\frac{a}{1-a}$ or -1 are negative since (2).

The new equation is $\left[x - \left(-\frac{1-a}{a}\right)^2\right]\left[x - \left(\frac{1}{-1}\right)^2\right] = 0$ or $a^2x^2 - (1 - 2a + 2a^2)x + (1 - 2a + a^2) = 0$

22. $y + 2 = \frac{\beta}{\gamma} + \frac{\gamma}{\beta} + 2 = \frac{\beta^2 + \gamma^2 + 2\beta\gamma}{\beta\gamma} = \frac{\alpha^2}{\beta\gamma} = \frac{\alpha^3}{\alpha\beta\gamma} = -\frac{\alpha^3}{q} = \frac{p\alpha + q}{q} = \frac{p\alpha}{q} + 1 = \frac{px}{q} + 1 \Rightarrow x = \frac{q(y+1)}{p}$

The new equation in y is

$x^3 + px + q = 0 \Rightarrow \left[\frac{q(y+1)}{p}\right]^3 + p\left[\frac{q(y+1)}{p}\right] + q = 0 \Rightarrow q^2y^3 + 3q^2y^2 + \left(p^3 + 3q^2\right)y + \left(2p^3 + q^2\right) = 0$

23. $\alpha + \beta = -\frac{b}{a}$ (1) , $\alpha\beta = \frac{c}{a}$ (2)

From (1), $a\alpha + b = \beta$, $a\beta + b = \alpha$ (3)

(a) $\dfrac{1}{(a\alpha+b)^2}+\dfrac{1}{(a\beta+b)^2}=\dfrac{1}{\beta^2}+\dfrac{1}{\alpha^2}=\dfrac{\alpha^2+\beta^2}{\alpha^2\beta^2}=\dfrac{(\alpha+\beta)^2-2\alpha\beta}{(\alpha\beta)^2}=\dfrac{\left(-\dfrac{b}{a}\right)^2-2\left(\dfrac{c}{a}\right)}{\left(\dfrac{c}{a}\right)^2}=\dfrac{b^2-2ac}{a^2c^2}$

(b) $\dfrac{1}{(a\alpha+b)^3}+\dfrac{1}{(a\beta+b)^3}=\dfrac{1}{\beta^3}+\dfrac{1}{\alpha^3}=\dfrac{\alpha^3+\beta^3}{\alpha^3\beta^3}=\dfrac{(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)}{(\alpha\beta)^3}=\dfrac{\left(-\dfrac{b}{a}\right)^3-3\left(\dfrac{c}{a}\right)\left(-\dfrac{b}{a}\right)}{\left(\dfrac{c}{a}\right)^3}=\dfrac{-b^3+3abc}{a^3c^3}$

24. $3\alpha^2+\alpha-1=0$ (1), $3\beta^2+\beta-1=0$ (2)

$(1)\times\alpha+(2)\times\beta$, $3(\alpha^3+\beta^3)+(\alpha^2+\beta^2)-(\alpha+\beta)=0$, $\alpha,\beta\neq 0$. (3)

$(1)\times\alpha^2+(2)\times\beta^2$, $3(\alpha^4+\beta^4)+(\alpha^3+\beta^3)-(\alpha^2+\beta^2)=0$. (4)

From (3), $\alpha^3+\beta^3=\dfrac{1}{3}\left[(\alpha+\beta)-(\alpha^2+\beta^2)\right]=\dfrac{1}{3}\left[(\alpha+\beta)-(\alpha+\beta)^2+2\alpha\beta\right]=-\dfrac{10}{27}$

From (4), $\alpha^4+\beta^4=\dfrac{1}{3}\left[(\alpha^2+\beta^2)-(\alpha^3+\beta^3)\right]=\dfrac{1}{3}\left[(\alpha+\beta)^2+2\alpha\beta+\dfrac{10}{27}\right]=\dfrac{31}{81}$

25. $a+b+c=-p$ (1) , $ab+bc+ca=q$ (2) , $abc=-r$ (3)

Since $x^3=-px^2-qx-r$

$a^3+b^3+c^3=-p(a^2+b^2+c^2)-q(a+b+c)-3r=-p\left[(a+b+c)^2-2(ab+bc+ca)\right]$

$=-p\left[(-p)^2-2q\right]-q(-p)-3r=-p^3+3pq-3r$ (4)

$\dfrac{1}{a},\dfrac{1}{b},\dfrac{1}{c}$ are roots of the equation : $\left(\dfrac{1}{x}\right)^3+p\left(\dfrac{1}{x}\right)+q\left(\dfrac{1}{x}\right)+r=0$ or $x^3+\dfrac{q}{r}x^2+\dfrac{p}{r}x+\dfrac{1}{r}=0$

From (4), $\dfrac{1}{a^3}+\dfrac{1}{b^3}+\dfrac{1}{c^3}=-\left(\dfrac{q}{r}\right)^3+3\left(\dfrac{q}{r}\right)\left(\dfrac{p}{r}\right)-3\left(\dfrac{1}{r}\right)=\dfrac{3pqr-q^3-3r^2}{r^3}$.

26. $x^2+2ax+b=0 \Rightarrow x=-a\pm\sqrt{a^2-b}$

$y^2+2py+q=0 \Rightarrow y=-p\pm\sqrt{p^2-q}$

To form the equation whose roots are the four numbers obtaining by adding the roots from each equation:

$z=x+y=-(a+p)\pm\sqrt{a^2-b}\pm\sqrt{p^2-q}\Rightarrow[z+(a+p)]^2=a^2-b\pm\sqrt{(a^2-b)(p^2-q)}+p^2-q$

$\Rightarrow z^2+2(a+p)z+b+q+2ap=\pm\sqrt{(a^2-b)(p^2-q)}$

Since z is dummy, we have $[x^2+2(a+p)x+b+q+2ap]^2-4(a^2-b)(p^2-q)=0$.

27. $a+b+c=0$ (1) $ab+bc+ca=m$ (2) $abc=-n$ (3)

$a^3 + ma + n = 0$ (4) $b^3 + mb + n = 0$ (5) $c^3 + cm + n = 0$ (6)

$(4) \times a^3 + (5) \times b^3 + (6) \times c^3$, $a^6 + b^6 + c^6 + m(a^4 + b^4 + c^4) + n(a^3 + b^3 + c^3) = 0$

$\therefore \quad a^6 + b^6 + c^6 = -m(a^4 + b^4 + c^4) - n(a^3 + b^3 + c^3)$ (7)

(4) + (5) + (6), $a^3 + b^3 + c^3 + m(a + b + c) + 3n = 0$

By (1), $a^3 + b^3 + c^3 + m \times 0 + 3n = 0 \quad \Rightarrow -n = \frac{1}{3}(a^3 + b^3 + c^3)$ (8)

$a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + bc + ca) = -2m \quad \Rightarrow -m = \frac{1}{2}(a^2 + b^2 + c^2)$ (9)

Substitute (8), (9) in (7), $a^6 + b^6 + c^6 = \frac{1}{3}(a^3 + b^3 + c^3)^2 + \frac{1}{2}(a^2 + b^2 + c^2)(a^4 + b^4 + c^4)$

28. If $1, x_1, x_2, x_3, x_4$ are the roots of the equation $x^5 - 1 = 0$, then

$0, 1 - x_1, 1 - x_2, 1 - x_3, 1 - x_4$ are the roots of the equation $(1 - x)^5 - 1 = 0$ or $(x + 1)^5 - 1 = 0$

or $x^5 - 5x^4 + 10x^3 - 10x^2 + 5x = 0$

$1 - x_1, 1 - x_2, 1 - x_3, 1 - x_4$ are the roots of the equation $x^4 - 5x^3 + 10x^2 - 10x + 5 = 0$

Product of roots = $(1 - x_1)(1 - x_2)(1 - x_3)(1 - x_4) = 5$

Similarly, $2, 1 + x_1, 1 + x_2, 1 + x_3, 1 + x_4$ are the roots of the equation $(x - 1)^5 - 1 = 0$

or $x^5 - 5x^4 + 10x^3 - 10x^2 + 5x - 2 = 0$

Product of roots = $2(1 + x_1)(1 + x_2)(1 + x_3)(1 + x_4) = 2$

$\therefore \quad (1 + x_1)(1 + x_2)(1 + x_3)(1 + x_4) = 1$

29. **Method 1**

$1, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ are roots of the equation $x^n = 1$.

$0, 1 - \alpha_1, 1 - \alpha_2, \ldots, 1 - \alpha_{n-1}$ are roots of the equation : $(1 - x)^n = 1$.

or $1 - nx + \frac{n(n-1)}{2}x^2 - \ldots + (-1)^n x^n = 1$ or $x\left[-n + \frac{n(n-1)}{2}x - \ldots + (-1)^n x^{n-1}\right] = 0$

$1 - \alpha_1, 1 - \alpha_2, \ldots, 1 - \alpha_{n-1}$ are roots of the equation : $(-1)^n x^{n-1} + \ldots + \frac{n(n-1)}{2}x - n = 0$

Product of roots = $(1 - \alpha_1)(1 - \alpha_2) \ldots (1 - \alpha_{n-1}) = n$.

Method 2

$1, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ are roots of the equation $x^n = 1$ (1)

$\therefore \quad (x - 1)(x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_{n-1}) = 0$ (2)

Differentiate (1) w.r.t. x, $nx^{n-1} = 0 \quad (x \neq 1)$ (3)

Differentiate (2) w.r.t. x, $(x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_{n-1}) + (x - 1)\,p(x) = 0$, where p(x) is a polynomial in x. (4)

Since (3) and (4) represent the same equation, we have

$nx^{n-1} = (x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_{n-1}) + (x - 1)\,p(x)$ (5)

Put $x = 1$ in (5), we have $(1 - \alpha_1)(1 - \alpha_2) \ldots (1 - \alpha_{n-1}) = n$.

Method 3

$x^n - 1 = (x - 1)(x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_{n-1})$

$x^{n-1} + x^{n-2} + \ldots + x \quad + 1 = (x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_{n-1})$

Put $x = 1$, we have $(1 - \alpha_1)(1 - \alpha_2) \ldots (1 - \alpha_{n-1}) = n$.

30. $x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_{n-1} x + p_n = (x - a_1)(x - a_2) \ldots (x - a_n)$ (1)

Sub. $x = i$ in (1), $i^n + p_1 i^{n-1} + p_2 i^{n-2} + \ldots + p_{n-1} i + p_n = (i - a_1)(i - a_2) \ldots (i - a_n)$ (2)

Sub. $x = -i$ in (1), $(-i)^n + p_1 (-i)^{n-1} + p_2(-i)^{n-2} + \ldots + p_{n-1}(-i) + p_n = (-i - a_1)(-i - a_2) \ldots (-i - a_n)$ (3)

$\therefore$ $(1 + a_1^2)(1 + a_2^2) \ldots (1 + a_n^2) = [(i - a_1)(-i - a_1)][(i - a_2)(-i - a_2)] \ldots [(i - a_n)(-i - a_n)]$

$= [(i - a_1)(i - a_2) \ldots (i - a_n)][(-i - a_1)(-i - a_2) \ldots (-i - a_n)]$

$= [i^n + p_1 i^{n-1} + p_2 i^{n-2} + \ldots + p_{n-1} i + p_n][(-i)^n + p_1(-i)^{n-1} + p_2(-i)^{n-2} + \ldots + p_{n-1}(-i) + p_n]$, by (2) , (3)

$= i^n [i^n + p_1 i^{n-1} + p_2 i^{n-2} + \ldots + p_{n-1} i + p_n] (-i)^n [(-i)^n + p_1(-i)^{n-1} + p_2(-i)^{n-2} + \ldots + p_{n-1}(-i) + p_n]$

$= [1 + p_1 i - p_2 - p_3 i + p_4 + p_5 i - \ldots][1 - p_1 i - p_2 + p_3 i + p_4 - p_5 i + \ldots]$

$= (1 - p_2 + p_4 - \ldots)^2 + (p_1 - p_3 + p_5 - \ldots)^2$.

31. Let $p(x) = x^3 + 3x - 3$, then $p(0) = -3 < 0$ and $p(1) = 1 > 0$.

Since there is a change of sign as x increases from 0 to 1, there is at least one real root α between $x = 0$ and $x = 1$ for $p(x) = 0$.

$p'(x) = 3x^2 + 3 > 0$. $\therefore$ $p(x)$ is increasing.

$\therefore$ $p(x) = 0$ has one and only one real α between $x = 0$ and $x = 1$.

Let $q(x) = x^4 + 6x^2 - 12x - 9$, then $q(2) > 0$, $q(1) < 0$ and $q(-1) > 0$.

$\therefore$ There is at least one real root between $x = -1$ and 1 and there is at least one root between $x = 1$ and 2.

However, $q'(x) = 4(x^3 + 3x - 3) = 4p(x)$.

Since $p(x) = 0$ has one and only one real root, $q'(x) = 0$ has one and only one real root.

$\therefore$ $q(x)$ has exactly one turning point and $q(x) = 0$ has just two real roots.

32. (a) Let $f(x)\ x^4 - 14x^2 + 24x - k$, then $f'(x) = 4x^3 - 28x + 24 = 4(x^3 - 7x + 6) = 4(x + 3)(x - 1)(x - 2)$

and the roots of $f'(x) = 0$ are $x = -3, 1$ and 2.

x	$-\infty$	-3	1	2	$+\infty$
f(x)	+	$-117 - k$	$11 - k$	$8 - k$	+

The equation will have four unequal real roots if the signs of $f(x)$ are alternate. This is so if

$-117 - k < 0$, $11 - k > 0$ and $8 - k < 0$.

These inequalities require that $8 < k < 11$.

(b) Let $g(x) = 3x^4 - 16x^3 + 6x^2 + 72x - k$ then $g'(x) = 12x^3 - 48x^2 + 16x + 72 = 12(x + 1)(x - 2)(x - 3)$

and the roots of $g'(x) = 0$ are $x = -1, 2$ and 3 .

x	$-\infty$	-1	2	3	$+\infty$
g(x)	+	$-47 - k$	$88 - k$	$81 - k$	+

The equation will have four unequal real roots if $-47 - k < 0$, $88 - k > 0$ and $81 - k < 0$.

These inequalities require that $81 < k < 88$.

33. **(a)** $p(x) = (x - a_1)(x - a_2)...(x - a_n)\sum_{r=1}^{n}\frac{1}{x - a_r}$

$= (x - a_2)(x - a_3) \dots (x - a_n) + (x - a_1)(x - a_3) \dots (x - a_n) + \dots + (x - a_1)(x - a_2) \dots (x - a_{n-1})$

Without lost of generality, let $a_1 < a_2 < \dots < a_n$.

If n is even, $p(a_1) = (a_1 - a_2)(a_1 - a_2) \dots (a_1 - a_n) < 0$, $p(a_2) = (a_2 - a_1)(a_2 - a_3) \dots (a_2 - a_n) > 0$,

$p(a_3) = (a_3 - a_1)(a_3 - a_2)(a_3 - a_4) \dots (a_3 - a_n) < 0, \dots , p(a_n) = (a_n - a_1)(a_n - a_3) \dots (a_n - a_{n-1}) > 0$

If n is odd, $p(a_1) > 0$, $p(a_2) < 0$, $p(a_3) > 0 , \dots, p(a_n) > 0$.

$\therefore$ $p(x)$ changes its sign $(n - 1)$ times as x increases.

Since $p(x)$ is continuous, therefore $p(x)$ has exactly $(n - 1)$ roots.

But $\sum_{r=1}^{n}\frac{1}{x - a_r} = 0$ and $p(x) = 0$ are equivalent equations since $x = a_1 , a_2, \dots, a_n$ are obviously not roots of both of the equations. Therefore $\sum_{r=1}^{n}\frac{1}{x - a_r} = 0$ has exactly $(n - 1)$ roots.

(b) Rolle's Theorem – Let f is continuous on $[a , b]$ and differentiable on (a, b) and if $f(a) = f(b)$, then there is a number $x_0 \in (a, b)$ such that $f'(x_0) = 0$.

Without lost of generality, let $a_1 < a_2 < \dots < a_n$.

Let $f(x) = (x - a_1) \dots (x - a_n)$.

Obviously $f(a_1) = f(a_2) = \dots = f(a_n) = 0$.

By Rolle's Theorem, there exists $x_i \in (a_i , a_{i+1})$, $i = 1, 2, \dots, (n - 1)$ such that $f'(x_i) = 0$.

$\therefore$ $x_1 , x_2 , \dots, x_{n-1}$ are roots of the equation $f'(x_i) = 0$.

But $f'(x) = f(x)\sum_{r=1}^{n}\frac{1}{x - a_r}$ $\therefore$ $x_1 , x_2 , \dots, x_{n-1}$ are roots of the equation $\sum_{r=1}^{n}\frac{1}{x - a_r} = 0$.

34. **(a)** If $p_n < 0$, and n is even, $p(x) \equiv x^n + p_1 x^{n-1} + p_2 x^{n-2} + \dots + p_{n-1} x + p_n = 0$

$\lim_{x\to+\infty} p(x) = +\infty$, $p(0) = p_n < 0$, $\lim_{x\to-\infty} p(x) = +\infty$.

There are 2 sign changes as x increases from $-\infty$ to $+\infty$.

$\therefore$ $p(x) = 0$ has at least one positive and at least one negative root.

(b) If $p_n < 0$, then $p(0) = p_n < 0$ and $\lim_{x\to+\infty} p(x) = +\infty$.

Since polynomials are continuous, the curve $y = p(x)$ cuts the x-axis odd number of times.

$\therefore$ The number of positive roots of $p(x) = 0$ is odd. (For any root of multiplicity k , we have to count k times)

If $p_n > 0$, then $p(0) = p_n > 0$ and $\lim_{x\to+\infty} p(x) = +\infty$.

The curve $y = p(x)$ cuts the x-axis even number of times. (or 0 number of times)

$\therefore$ The number of positive roots of $p(x) = 0$ is even, or zero.